TEA013 Matemática Aplicada II
Curso de Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
F, 20 Dez 2024
Prof. Nelson Luís Dias

0

NOME: GABARITO Assinatura: ______________

AO REALIZAR ESTA PROVA, VOCÊ DEVE JUSTIFICAR TODAS AS PASSAGENS. EVITE "PULAR" PARTES IMPORTANTES DO DESENVOLVIMENTO DE CADA QUESTÃO. JUSTIFIQUE CADA PASSO IMPORTANTE. SIMPLIFIQUE AO MÁXIMO SUAS RESPOSTAS.

ATENÇÃO PARA A NOTAÇÃO VETORIAL E TENSORIAL! VETORES MANUSCRITOS DEVEM SER ESCRITOS COMO $\underset{\sim}{v}$; TENSORES DE ORDEM 2 COMO $\underset{\approx}{A}$.

---

1 [20] **Sem utilizar frações parciais**, encontre a transformada de Laplace inversa

$$\mathscr{L}^{-1}\left\{\frac{s}{(s-3)(s^2+1)}\right\}.$$

SOLUÇÃO DA QUESTÃO:
A expressão acima é o produto de duas transformadas de Laplace conhecidas:

$$\begin{aligned}\mathscr{L}\left\{\mathrm{e}^{3t}\right\} &= \frac{1}{s-3},\\ \mathscr{L}\left\{\cos(t)\right\} &= \frac{s}{s^2+1}.\end{aligned}$$

Pelo Teorema da Convolução,

$$\begin{aligned}\mathscr{L}\left\{[f*g](t)\right\} &= \overline{f}(s)\overline{g}(s),\\ &= \frac{1}{s-3}\times\frac{s}{s^2+1} \Rightarrow\\ \mathscr{L}^{-1}\left\{\frac{s}{(s-3)(s^2+1)}\right\} &= \mathrm{e}^{3t}*\cos(t)\\ &= \int_{\tau=0}^{t}\mathrm{e}^{3(t-\tau)}\cos(\tau)\,\mathrm{d}\tau\\ &= \operatorname{Re}\left[\mathrm{e}^{3t}\int_{\tau=0}^{t}\mathrm{e}^{-3\tau}\mathrm{e}^{\mathrm{i}\tau}\,\mathrm{d}\tau\right] = \operatorname{Re}\left[\mathrm{e}^{3t}\int_{\tau=0}^{t}\mathrm{e}^{(\mathrm{i}-3)\tau}\,\mathrm{d}\tau\right]\\ &= \operatorname{Re}\left[\mathrm{e}^{3t}\frac{1}{\mathrm{i}-3}\int_{\tau=0}^{t}\mathrm{e}^{(\mathrm{i}-3)\tau}\,\mathrm{d}(\mathrm{i}-3)\tau\right]\\ &= \operatorname{Re}\left\{\frac{\mathrm{e}^{3t}}{\mathrm{i}-3}\mathrm{e}^{(\mathrm{i}-3)\tau}\Big|_{\tau=0}^{t}\right\} = \operatorname{Re}\left\{\frac{\mathrm{e}^{3t}}{\mathrm{i}-3}\left[\mathrm{e}^{(\mathrm{i}-3)t}-1\right]\right\}\\ &= \operatorname{Re}\left\{\frac{1}{\mathrm{i}-3}\left[\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{3t}\right]\right\} = \operatorname{Re}\left\{\frac{-\mathrm{i}-3}{10}\left[\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{3t}\right]\right\}\\ &= \frac{1}{10}\operatorname{Re}\left\{-\mathrm{i}\mathrm{e}^{\mathrm{i}t}+\mathrm{i}\mathrm{e}^{3t}-3\mathrm{e}^{\mathrm{i}t}+3\mathrm{e}^{3t}\right\}\\ &= \frac{1}{10}\left[\operatorname{sen}(t)+0-3\cos(t)+3\mathrm{e}^{3t}\right] \blacksquare\end{aligned}$$

Continue a solução no verso ⟹

2 [20] Obtenha

$$\int_{-\infty}^{x} H(\xi - a)\cos(\xi)\,\mathrm{d}\xi$$

onde $H(x)$ é a função de Heaviside.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
\int_{-\infty}^{x} \underbrace{H(\xi - a)}_{u}\,\underbrace{\cos(\xi)\,\mathrm{d}\xi}_{\mathrm{d}v} &= H(\xi - a)\,\mathrm{sen}(\xi)\Big|_{-\infty}^{x} - \int_{-\infty}^{x} \mathrm{sen}(\xi)\delta(\xi - a)\,\mathrm{d}\xi \\
&= H(x - a)\,\mathrm{sen}(x) - H(x - a)\,\mathrm{sen}(a) \\
&= H(x - a)[\mathrm{sen}(x) - \mathrm{sen}(a)] \;\blacksquare
\end{aligned}$$

Continue a solução no verso $\Longrightarrow$

3 [20] Aplique a desigualdade de Schwarz para dois vetores $\boldsymbol{u}$, $\boldsymbol{v}$ do $\mathbb{R}^3$ tais que

$$\begin{aligned} \boldsymbol{u} &= (x, y, z), \qquad \text{onde} \qquad x^2 + y^2 + z^2 = 1, \\ \boldsymbol{v} &= (1, 2, 3), \end{aligned}$$

utilizando o produto escalar padrão. **Simplifique ao máximo.**

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned} (\boldsymbol{u} \cdot \boldsymbol{v})^2 &\le (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) \\ (x + 2y + 3z)^2 &\le (x^2 + y^2 + z^2)(1 + 4 + 9) \\ (x + 2y + 3z)^2 &\le 14 \;\blacksquare \end{aligned}$$

Continue a solução no verso $\Longrightarrow$

4 [20] Obtenha a série de Fourier **complexa** de

$$f(x) = \mathrm{e}^{-x}, \qquad 0 \le x \le 1.$$

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{\frac{2ni\pi x}{L}};\\
c_n &= \frac{1}{L}\int_a^b \mathrm{e}^{-\frac{2ni\pi x}{L}} f(x)\,\mathrm{d}x;\\
a &= 0,\\
b &= 1,\\
L &= b - a = 1;\\
c_n &= \int_0^1 \mathrm{e}^{-x}\mathrm{e}^{-2ni\pi x}\,\mathrm{d}x\\
&= -\frac{1}{\mathrm{e}}\frac{(\mathrm{e}-1)(2\mathrm{i}\pi n - 1)}{4\pi^2 n^2 + 1} \blacksquare
\end{aligned}$$

Continue a solução no verso ⟹

5 [20] Obtenha a função de Green da equação diferencial

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x^2 y = f(x), \qquad y(0) = 1.$$

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}\xi} - \xi^2 y &= f(\xi),\\
G(x,\xi)\frac{\mathrm{d}y}{\mathrm{d}\xi} - G(x,\xi)\xi^2 y &= G(x,\xi)f(\xi),\\
\int_0^\infty G(x,\xi)\frac{\mathrm{d}y}{\mathrm{d}\xi}\,\mathrm{d}\xi - \int_0^\infty G(x,\xi)\xi^2 y\,\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi,\\
[G(x,\xi)y(\xi)]_0^\infty - \int_0^\infty y\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} - \int_0^\infty G(x,\xi)\xi^2 y\,\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi,\\
[G(x,\infty)y(\infty) - G(x,0)y(0)] - \int_0^\infty y\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} - \int_0^\infty G(x,\xi)\xi^2 y\,\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi,
\end{aligned}$$

faça

$$\begin{aligned}
G(x,\infty) &= 0; \Rightarrow\\
-G(x,0) + \int_{\xi=0}^\infty \left[-\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} - \xi^2 G(x,\xi)\right] y(\xi)\,\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

A equação diferencial em G é

$$\begin{aligned}
-\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} - \xi^2 G(x,\xi)y(\xi) &= \delta(\xi - x),\\
\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} + \xi^2 G(x,\xi)y(\xi) &= -\delta(\xi - x),\\
G(x,\xi) &= u(x,\xi)v(x,\xi),\\
u\frac{\mathrm{d}v}{\mathrm{d}\xi} + v\frac{\mathrm{d}u}{\mathrm{d}\xi} + \xi^2 uv &= -\delta(\xi - x),\\
u\left[\frac{\mathrm{d}v}{\mathrm{d}\xi} + \xi^2 v\right] + v\frac{\mathrm{d}u}{\mathrm{d}\xi} &= -\delta(\xi - x),\\
\frac{\mathrm{d}v}{v} &= -\xi^2\,\mathrm{d}\xi,\\
\int_{v(x,0)}^{v(x,\xi)} \frac{\mathrm{d}v}{v} &= -\int_{\eta=0}^{\xi} \eta^2\,\mathrm{d}\eta\\
\ln\left(\frac{v(x,\xi)}{v(x,0)}\right) &= -\frac{\xi^3}{3}\\
v(x,\xi) &= v(x,0)\exp\left(-\frac{\xi^3}{3}\right);\\
v(x,0)\exp\left(-\frac{\xi^3}{3}\right)\frac{\mathrm{d}u}{\mathrm{d}\xi} &= -\delta(\xi - x),\\
\frac{\mathrm{d}u}{\mathrm{d}\eta} &= -\frac{1}{v(x,0)}\exp\left(\frac{\eta^3}{3}\right)\delta(\eta - x),\\
\int_{u(x,0)}^{u(x,\xi)} \mathrm{d}u &= -\int_{\eta=0}^{\xi} \frac{1}{v(x,0)}\exp\left(\frac{\eta^3}{3}\right)\delta(\eta - x)\,\mathrm{d}\eta,\\
u(x,\xi) &= u(x,0) - \frac{1}{v(x,0)}H(\xi - x)\exp\left(\frac{x^3}{3}\right).
\end{aligned}$$

Continue a solução no verso ⟹

Obtemos, para $G(x,\xi)$,

$$
\begin{aligned}
G(x,\xi) &= u(x,\xi)v(x,\xi)\\
&= \left[u(x,0) - \frac{1}{v(x,0)}H(\xi - x)\exp\left(\frac{x^3}{3}\right)\right] v(x,0)\exp\left(-\frac{\xi^3}{3}\right)\\
&= G(x,0)\exp\left(-\frac{\xi^3}{3}\right) - H(\xi - x)\exp\left(\frac{x^3}{3}\right)\exp\left(-\frac{\xi^3}{3}\right)\\
&= \exp\left(-\frac{\xi^3}{3}\right)\left[G(x,0) - H(\xi - x)\exp\left(\frac{x^3}{3}\right)\right].
\end{aligned}
$$

Mas

$$
\begin{aligned}
G(x,\infty) &= 0,\\
G(x,0) - \exp\left(\frac{x^3}{3}\right) &= 0,\\
G(x,0) &= \exp\left(\frac{x^3}{3}\right);
\end{aligned}
$$

finalmente,

$$
G(x,\xi) = [1 - H(\xi - x)]\exp\left(\frac{x^3}{3}\right)\exp\left(-\frac{\xi^3}{3}\right) \blacksquare
$$

Continue a solução no verso $\Longrightarrow$